1.1 Solving Problems by Inductive Reasoning

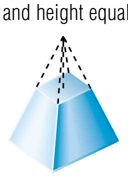
The Moscow papyrus, which dates back to about 1850 B.C., provides an example of inductive reasoning by the early Egyptian mathematicians. Problem 14 in the document reads:

> *You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take one-third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.*

What does all this mean? A *frustum* of a pyramid is that part of the pyramid remaining after its top has been cut off by a plane parallel to the base of the pyramid. The actual formula for finding the volume of the frustum of a pyramid with a square base is

$$V = \frac{1}{3}h(b^2 + bB + B^2),$$

where b is the area of the upper base, B is the area of the lower base, and h is the height (or altitude). The writer of the problem is giving a method of determining the volume of the frustum of a pyramid with square bases on the top and bottom, with bottom base side of length 4, top base side of length 2, and height equal to 6.



A truncated pyramid, or frustum of a pyramid

The development of mathematics can be traced to the Egyptian and Babylonian cultures (3000 B.C.–A.D. 260) as a necessity for problem solving. Their approach was an example of the "do thus and so" method: in order to solve a problem or perform an operation, a cookbook-like recipe was given, and it was performed over and over to solve similar problems. The classical Greek period (600 B.C.–A.D. 450) gave rise to a more formal type of mathematics, in which general concepts were applied to specific problems, resulting in a structured, logical development of mathematics.

By observing that a specific method worked for a certain type of problem, the Babylonians and the Egyptians concluded that the same method would work for any similar type of problem. Such a conclusion is called a *conjecture*. A **conjecture** is an educated guess based upon repeated observations of a particular process or pattern. The method of reasoning we have just described is called *inductive reasoning.*

Inductive Reasoning

Inductive reasoning is characterized by drawing a general conclusion (making a conjecture) from repeated observations of specific examples. The conjecture may or may not be true.

In testing a conjecture obtained by inductive reasoning, it takes only one example that does not work in order to prove the conjecture false. Such an example is called a **counterexample.** Inductive reasoning provides a powerful method of drawing conclusions, but it is also important to realize that there is no assurance that the observed conjecture will always be true. For this reason, mathematicians are reluctant to accept a conjecture as an absolute truth until it is formally proved using methods of *deductive reasoning.* Deductive reasoning characterized the development and approach of Greek mathematics, as seen in the works of Euclid, Pythagoras, Archimedes, and others.

Deductive Reasoning

Deductive reasoning is characterized by applying general principles to specific examples.

Let us now look at examples of these two types of reasoning. In this chapter we will often refer to the **natural** or **counting numbers:**

$$1, 2, 3, \ldots.$$

The three dots indicate that the numbers continue indefinitely in the pattern that has been established. The most probable rule for continuing this pattern is "add 1 to the previous number," and this is indeed the rule that we follow. Now consider the following list of natural numbers:

$$2, 9, 16, 23, 30.$$

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What is the next number of this list? Most people would say that the next number is 37. Why? They probably reason something like this: What have 2 and 9 and 16 in common? What is the pattern?

After studying the numbers, we might see that $2 + 7 = 9$, and $9 + 7 = 16$. Is something similar true for the other numbers in this list? Do you add 16 and 7 to get 23? Do you add 23 and 7 to get 30? Yes; any number in the given list can be found by adding 7 to the preceding number, so the next number in the list should be $30 + 7 = 37$.

You set out to find the "next number" by reasoning from your observation of the numbers in the list. You may have jumped from these observations above ($2 + 7 = 9$, $9 + 7 = 16$, and so on) to the general statement that any number in the list is 7 more than the preceding number. This is an example of *inductive reasoning.*

By using inductive reasoning, we concluded that 37 was the next number in the list. But this is wrong. You were set up. You've been tricked into drawing an incorrect conclusion. Not that your logic was faulty; but the person making up the list has another answer in mind. The list of numbers

$$2, 9, 16, 23, 30$$

actually gives the dates of Mondays in June if June 1 falls on a Sunday. The next Monday after June 30 is July 7. With this pattern, the list continues as

$$2, 9, 16, 23, 30, 7, 14, 21, 28, \ldots.$$

See the calendar in Figure 1.

June

S	M	Tu	W	Th	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30					

July

S	M	Tu	W	Th	F	S
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

FIGURE 1

The process you may have used to obtain the rule "add 7" in the list above reveals one main flaw of inductive reasoning. You can never be sure that what is true in a specific case will be true in general. Even a larger number of cases may not be enough. Inductive reasoning does not guarantee a true result, but it does provide a means of making a conjecture.

With deductive reasoning, we use general statements and apply them to specific situations. For example, one of the best-known rules in mathematics is the Pythagorean Theorem: In any right triangle, the sum of the squares of the legs (shorter sides) is equal to the square of the hypotenuse (longest side). Thus, if we know that the lengths of the shorter sides are 3 inches and 4 inches, we can find the length of the longest side. Let h represent the longest side:

$$\begin{aligned} 3^2 + 4^2 &= h^2 && \text{Pythagorean Theorem} \\ 9 + 16 &= h^2 && \text{Square the terms.} \\ 25 &= h^2 && \text{Add.} \\ h &= 5. && \text{The positive square root of 25 is 5.} \end{aligned}$$

Thus, the longest side measures 5 inches. We used the general rule (the Pythagorean Theorem) and applied it to the specific situation.

Reasoning through a problem usually requires certain *premises.* A **premise** can be an assumption, law, rule, widely held idea, or observation. Then reason inductively or deductively from the premises to obtain a **conclusion.** The premises and conclusion make up a **logical argument.**

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EXAMPLE 1 Identify each premise and the conclusion in each of the following arguments. Then tell whether each argument is an example of inductive or deductive reasoning.

(a) Our house is made of redwood. Both of my next-door neighbors have redwood houses. Therefore, all houses in our neighborhood are made of redwood.

The premises are "Our house is made of redwood" and "Both of my next-door neighbors have redwood houses." The conclusion is "Therefore, all houses in our neighborhood are made of redwood." Since the reasoning goes from specific examples to a general statement, the argument is an example of inductive reasoning (although it may very well have a false conclusion).

(b) All word processors will type the symbol @. I have a word processor. I can type the symbol @.

Here the premises are "All word processors will type the symbol @" and "I have a word processor." The conclusion is "I can type the symbol @." This reasoning goes from general to specific, so deductive reasoning was used.

(c) Today is Friday. Tomorrow will be Saturday.

There is only one premise here, "Today is Friday." The conclusion is "Tomorrow will be Saturday." The fact that Saturday follows Friday is being used, even though this fact is not explicitly stated. Since the conclusion comes from general facts that apply to this special case, deductive reasoning was used. ■

The example involving dates earlier in this section illustrated how inductive reasoning may, at times, lead to false conclusions. However, in many cases inductive reasoning does provide correct results, if we look for the most *probable* answer.

EXAMPLE 2 Use inductive reasoning to determine the *probable* next number in each list below.

(a) 3, 7, 11, 15, 19, 23

Each number in the list is obtained by adding 4 to the previous number. The probable next number is $23 + 4 = 27$.

(b) 1, 1, 2, 3, 5, 8, 13, 21

Beginning with the third number in the list, each number is obtained by adding the two previous numbers in the list. That is, $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on. The probable next number in the list is $13 + 21 = 34$. (These are the first few terms of the famous *Fibonacci sequence,* covered in detail in a later chapter.)

(c) 1, 2, 4, 8, 16

It appears here that in order to obtain each number after the first, we must double the previous number. Therefore, the most probable next number is $16 \times 2 = 32$. ■

Inductive reasoning often can be used to predict an answer in a list of similarly constructed computation exercises, as shown in the next example.

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$37 \times 3 = 111$
$37 \times 6 = 222$
$37 \times 9 = 333$
$37 \times 12 = 444$

EXAMPLE 3 Consider the list of equations in the margin. Use the list to predict the next multiplication fact in the list.

In each case, the left side of the equation has two factors, the first 37 and the second a multiple of 3, beginning with 3. The product (answer) in each case consists of three digits, all the same, beginning with 111 for 37×3. For this pattern to continue, the next multiplication fact would be $37 \times 15 = 555$, which is indeed true. (Note: You might wish to investigate what happens after 30 is reached for the right-hand factor, and make conjectures based on those products.)

FOR FURTHER THOUGHT

The following anecdote concerning inductive reasoning appears in the first volume of the *In Mathematical Circles* series by Howard Eves (PWS-KENT Publishing Company).

A scientist had two large jars before him on the laboratory table. The jar on his left contained a hundred fleas; the jar on his right was empty. The scientist carefully lifted a flea from the jar on the left, placed the flea on the table between the two jars, stepped back, and in a loud voice said, "Jump." The flea jumped and was put in the jar on the right. A second flea was carefully lifted from the jar on the left and placed on the table between the two jars. Again the scientist stepped back and in a loud voice said, "Jump." The flea jumped and was put in the jar on the right. In the same manner, the scientist treated each of the hundred fleas in the jar on the left, and each flea jumped as ordered. The two jars were then interchanged and the experiment continued with a slight difference. This time the scientist carefully lifted a flea from the jar on the left, yanked off its hind legs, placed the flea on the table between the jars, stepped back, and in a loud voice said, "Jump." The flea did not jump, and was put in the jar on the right. A second flea was carefully lifted from the jar on the left, its hind legs yanked off, and then placed on the table between the two jars. Again the scientist stepped back and in a loud voice said, "Jump." The flea did not jump, and was put in the jar on the right. In this manner, the scientist treated each of the hundred fleas in the jar on the left, and in no case did a flea jump when ordered. So the scientist recorded in his notebook the following induction: "A flea, if its hind legs are yanked off, cannot hear."

For Group Discussion

As a class, discuss examples from advertising on television, in newspapers, magazines, etc., that lead consumers to draw incorrect conclusions.

A classic example of the pitfalls in inductive reasoning involves the maximum number of regions formed when chords are constructed in a circle. When two points on a circle are joined with a line segment, a *chord* is formed. Locate a single point on a circle. Since no chords are formed, a single interior region is formed.

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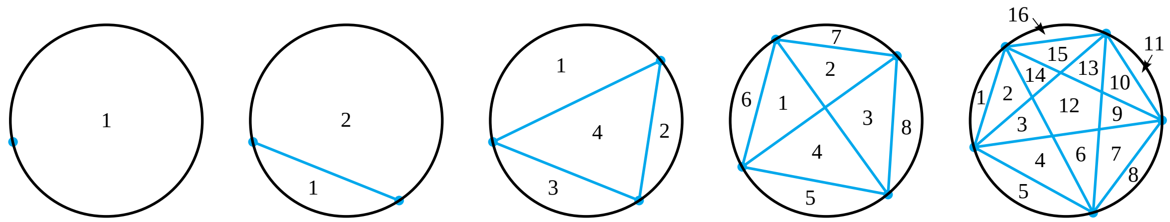
See Figure 2(a). Locate two points and draw a chord. Two interior regions are formed, as shown in Figure 2(b). Continue this pattern. Locate three points, and draw all possible chords. Four interior regions are formed, as shown in Figure 2(c). Four points yield 8 regions and five points yield 16 regions. See Figures 2(d) and 2(e).



(a) (b) (c) (d) (e)

FIGURE 2

Number of Points	Number of Regions
1	1
2	2
3	4
4	8
5	16

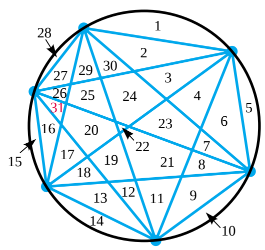


FIGURE 3

The results of the preceding observations are summarized in the chart in the margin. The pattern formed in the column headed "Number of Regions" is the same one we saw in Example 2(c), where we predicted that the next number would be 32. It seems here that for each additional point on the circle, the number of regions doubles. A reasonable inductive conjecture would be that for six points, 32 regions would be formed. But as Figure 3 indicates, there are only 31 regions!

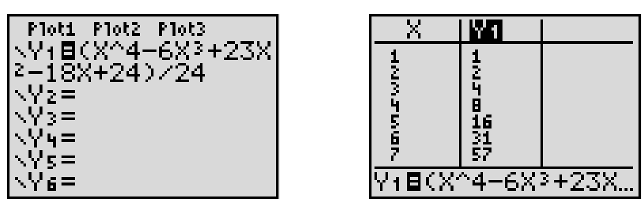
No, a region was not "missed." It happens that the pattern of doubling ends when the sixth point is considered. Adding a seventh point would yield 57 regions. The numbers obtained here are

$$1, 2, 4, 8, 16, 31, 57.$$

For n points on the circle, the number of regions is given by the formula

$$\frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}.*$$

We can use a graphing calculator to construct a table of values that indicates the number of regions for various numbers of points. Using X rather than n, we can define Y_1 using the expression above (see Figure 4(a)). Then, creating a table of values, as in Figure 4(b), we see how many regions (indicated by Y_1) there are for any number of points (X).



(a) (b)

FIGURE 4

*For more information on this and other similar patterns, see "Counting Pizza Pieces and Other Combinatorial Problems," by Eugene Maier, in the January 1988 issue of *Mathematics Teacher*, pp. 22–26.

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As indicated earlier, not until a general relationship is proved can one be sure about a conjecture since one counterexample is always sufficient to make the conjecture false.

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